

THE FIRST TERM OF PLETHYSMS

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ABSTRACT. Plethysm of two Schur functions can be expressed as a linear combination of Schur functions, and monomial symmetric functions. In this paper, we express the coefficients combinatorially in the case of monomial symmetric functions. And by using it, we determine the first term of the plethysm with respect to Schur functions under the reverse lexicographic order.

1. INTRODUCTION

Let λ and μ be partitions of positive integers m and n , respectively. The plethysm $s_\lambda[s_\mu]$ is the symmetric function obtained by substituting the monomials in s_μ for the variables of s_λ . D.E.Littlewood introduced this operation in 1936[9].

Plethysm of two Schur functions is expressed as a linear combination of Schur functions;

$$s_\lambda[s_\mu] = \sum_{\nu \vdash mn} a_{\lambda[\mu]}^\nu s_\nu ,$$

where $\nu \vdash mn$ means that ν is a partition of mn .

Plethysm appear in some fields, especially representation theory. For example, the coefficient $a_{\lambda[\mu]}^\nu$ is equal to the multiplicity of the irreducible GL_N -module of highest weight ν in a certain GL_N -module, and the multiplicity of the irreducible S_{mn} -module type of ν in a certain $S_n \wr S_m$ -module, where GL_N , S_{mn} , and $S_n \wr S_m$ are the general linear group, the symmetric group, and the wreath product of the symmetric group, respectively. (See [10;Chapter.1 Appendix A and Appendix B].) From these interpretations, we see that each of the coefficient $a_{\lambda[\mu]}^\nu$ is a nonnegative integer.

One of most fundamental problems for plethysm is expressing the coefficients $a_{\lambda[\mu]}^\nu$ combinatorially like Kostka coefficients and Littlewood-Richardson coefficients. In the case of $\lambda = (2)$ or (1^2) , Carré and Leclerc [4] found a combinatorial description by using domino tableaux. However it is generally open problem.

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In [2] Agaoka gave the table of $a_{\lambda[\mu]}^\nu$ up to $mn = 16$, and in [11] another method for calculating $a_{\lambda[\mu]}^\nu$ is given.

Moreover plethysm is a necessary tool when we consider some geometric problems. For example, in [1] Agaoka found a new obstruction of local isometric imbeddings of Riemannian submanifolds with codimension 2 by calculating the plethysm $s_3[s_{2,2}]$.

Now we explain our approach briefly. Plethysm is also expressed as a sum of monomial symmetric functions;

$$s_\lambda[s_\mu] = \sum_{\nu \vdash mn} Y_{\lambda[\mu]}^\nu m_\nu$$

where m_ν denotes the monomial symmetric function corresponding to partition ν . We give a combinatorial description of the coefficients $Y_{\lambda[\mu]}^\nu$ (see section 3), and using this, we determine the *first* term of $s_\lambda[s_\mu]$ (see section 4). (The first term is the maximal element among the partition ν satisfying $a_{\lambda[\mu]}^\nu \neq 0$ with respect to the reverse lexicographic order.)

2. PRELIMINARY

2.1. Young tableaux, symmetric functions. We start with introducing the notations. For a positive integer m , let $[1, m] = \{i \in \mathbb{Z} | 1 \leq i \leq m\}$ be the interval of integers between 1 and m . For a positive integer n , a *partition* of n is a non-increasing sequences of non-negative integers summing to n . We write $\lambda \vdash n$ if λ is a partition of size n . And we use the same notation λ to represent the Young diagram corresponding to λ . Let s_λ, m_λ , and h_λ denote Schur function, monomial symmetric function, and complete symmetric function corresponding to λ , respectively. Here we use x_1, x_2, \dots as variables. And we define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the ring of symmetric functions as follows;

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}.$$

Next we introduce notations for Young tableaux. For a given Young diagram λ , a *Young tableau* (of shape μ) is a map from the set of cells (in the Young diagram λ) to a totally ordered set S . For a given Young tableau T , the image of (i, j) is denoted by $T(i, j)$ and called the (i, j) *entry* of T . A *semi standard tableau* is a Young tableau whose entries increase weakly along the rows and increase strictly down the columns. For a Young diagram λ , $\text{SSTab}(\lambda, S)$ denotes the set of semi standard tableaux of shape λ .

In particular we can take the set of positive integers as a totally ordered set S . In this case we write $\text{SSTab}(\lambda, [1, m])$ simply $\text{SSTab}(\lambda)_{\leq m}$. For a Young tableau T , the *weight* of T is the sequence $\text{wt}(T) =$

(μ_1, μ_2, \dots) , where μ_k is the number of $T(i, j)$ equal to k . We denote by $\text{SSTab}(\lambda; \mu)$ the set of semi standard tableaux of shape λ with weight μ . For a tableau $T \in \text{SSTab}(\lambda; \mu)$, we define $x^T = x^{\text{wt}(T)} = x_1^{\mu_1} x_2^{\mu_2} \dots$.

Next we define a total order in $\text{SSTab}(\lambda)$ which is used in section 3. For a given semi standard tableau T , by reading T from left to right in consecutive rows, starting from the top to bottom, we obtain the *word* $\text{word}(T)$. We define a total order $>$ on the set of words (in which entry is a positive integer) as the lexicographic order.

Definition 2.1. *Let $T, U \in \text{SSTab}(\lambda)$. We define $T > U$ if $\text{word}(T) > \text{word}(U)$.*

Example 2.2. Let $T_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & & \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 4 & & \\ \hline \end{array}$, $T_3 = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & & \\ \hline \end{array}$. Then $\text{word}(T_1) = 1123$, $\text{word}(T_2) = 1124$, $\text{word}(T_3) = 1222$. Thus $T_1 < T_2 < T_3$.

Now we recall well-known results for Kostka coefficients.

Definition 2.3. *For $\lambda, \mu \vdash n$, the Kostka coefficient $K_{\lambda, \mu}$ is defined by*

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu} m_\mu.$$

Similarly the inverse Kostka coefficient $K_{\lambda, \mu}^{-1}$ is given by

$$m_\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu}^{-1} s_\mu.$$

By a simple consideration, we have the following;

Proposition 2.4.

$$h_\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu} s_\mu.$$

Next theorem supply us with a combinatorially expression of Kostka coefficients.

Theorem 2.5. *Let $\lambda, \mu \vdash n$, then we have*

$$K_{\lambda, \mu} = \#\text{SSTab}(\lambda; \mu).$$

Remark. For $K_{\lambda, \mu}^{-1}$ we also have a combinatorially expression.[5]

From this theorem we have some corollaries which we will use later.

Corollary 2.6. *We introduce a total order on the set of Young diagrams by the reverse lexicographic order.(see [10]) Then for $\lambda, \mu \vdash n$,*

(i) *If $\lambda < \mu$, then $K_{\lambda, \mu} = 0$.*

(ii) If $\lambda < \mu$, then $K_{\lambda,\mu}^{-1} = 0$.

For a positive integer m , by putting $x_{m+1} = x_{m+2} = \cdots = 0$ in the theorem, we have the next corollary.

Corollary 2.7. *For a Schur function with m variables $s_\lambda(x_1, \dots, x_m)$, we have*

$$s_\lambda(x_1, \dots, x_m) = \sum_{T \in \text{SSTab}(\mu)_{\leq m}} x^T.$$

2.2. plethysm. Let f and g be two symmetric functions and write g as a sum of monomials: $g = \sum_{\alpha \in \mathbb{N}^\infty} c_\alpha x^\alpha$. Introduce the set of fictitious variables y_i defined by

$$\Pi(1 + y_i t) = \Pi_{\alpha \in \mathbb{N}^\infty} (1 + x^\alpha t)^{c_\alpha}$$

and define $f[g] = f(y_1, y_2, \dots)$. If f is n -th symmetric function and g is m -th, then $f[g]$ is nm -th symmetric function. We call this multiple on the set of symmetric functions *plethysm*.

Proposition 2.8. *Let f and g be two symmetric functions. We restrict g to s -variables and write it as a sum of monomials:*

$$g(x_1, \dots, x_s, 0, 0, \dots) = \sum_{i=1}^N x^{\alpha(i)}.$$

Then,

$$f[g](x_1, \dots, x_s, 0, 0, \dots) = f(x^{\alpha(1)}, \dots, x^{\alpha(N)}, 0, 0, \dots).$$

That is, $f[g](x_1, \dots, x_s, 0, 0, \dots)$ is the symmetric polynomial obtained by substituting monomials in f (together with multiplicity) for the variables in g .

3. THE EXPRESSION OF PLETHYSM IN MONOMIAL SYMMETRIC FUNCTIONS

Let $\lambda \vdash m$, $\mu \vdash n$ and $\nu \vdash mn$. We put a copy the Young diagram μ in each cell of the Young diagram λ , and denote such a diagram by $\lambda[\mu]$. For example, if $\lambda = (3, 1)$ and $\mu = (3, 2)$, we consider the following diagram. (Fig.1)

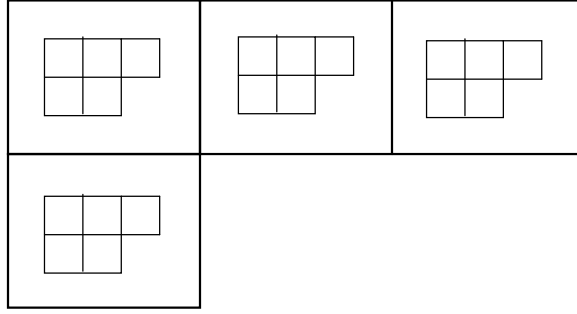


Fig.1

Definition 3.1. A semi standard tableau of shape $\lambda[\mu]$ is a semi standard tableau $T : \lambda \rightarrow \text{SSTab}(\mu)$ in the sense of Definition 2.1. Namely it is filled with mn number of positive integers and it satisfies following two conditions;

- (i). Each Young tableau of shape μ is a semi standard tableau.
- (ii). These m number of semi standard tableaux form a semi standard tableau of shape λ with respect to the totally order in Definition 2.1.

Moreover $\text{SSTab}(\lambda[\mu])$ denotes the set of semi standard tableaux of shape $\lambda[\mu]$.

Definition 3.2. For given $T \in \text{SSTab}(\lambda[\mu])$ we define the weight $\text{wt}(T)$ as usual, i.e. $\text{wt}(T) = (\nu_1, \nu_2, \dots)$, where ν_k is the number of entries equal to k . For $\lambda \vdash m, \mu \vdash n$ and $\nu \vdash mn$, we put

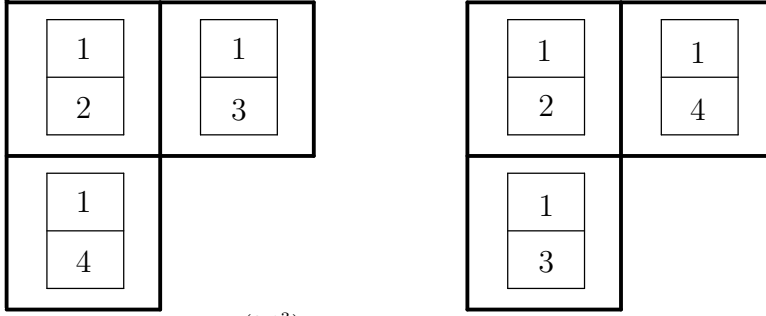
$$Y_{\lambda[\mu]}^{\nu} := \#\text{SSTab}(\lambda[\mu]; \nu).$$

Example 3.3. Set $\lambda = (2), \mu = (2)$ and $\nu = (2, 1, 1)$. Then the Young tableaux of shape $(2)[(2)]$ with weight $(2, 1, 1)$ are as follows;



Hence we have $Y_{(2)[(2)]}^{(2,1^2)} = 2$.

Example 3.4. Set $\lambda = (2, 1), \mu = (1^2)$ and $\nu = (3, 1^3)$. Then the Young tableaux of shape $(2, 1)[(1^2)]$ with weight $(3, 1^3)$ are as follows;



Hence we have $Y_{(2,1)[(1^2)]}^{(3,1^3)} = 2$.

Now we prove the first main result in this paper.

Theorem 3.5. *Let $\lambda \vdash m, \mu \vdash n$ and $\nu \vdash mn$. Then $Y_{\lambda[\mu]}^\nu$ is equal to the coefficient of m_ν in the expansion of $s_\lambda[s_\mu]$ in terms of monomial symmetric functions.*

In other words, we have

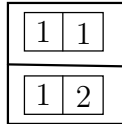
$$s_\lambda[s_\mu] = \sum_{\nu \vdash mn} Y_{\lambda[\mu]}^\nu m_\nu.$$

Proof. Before the proof, we introduce some notations. For a positive integer s , we set $r = \#\text{SSTab}(\mu)_{\leq s}$. For $1 \leq i \leq r$, let T_i be the i -th largest semi standard tableau in $\text{SSTab}(\mu)_{\leq s}$ and set $y_i = x^{T_i}$. In particular, $\text{SSTab}(\mu)_{\leq s} = \{T_1, \dots, T_r\}$. Note that there is a natural bijection

$$\iota: \text{SSTab}(\lambda)_{\leq r} \longrightarrow \text{SSTab}(\lambda, \text{SSTab}(\mu)_{\leq s}) \longrightarrow \text{SSTab}(\lambda[\mu])_{\leq s}$$

such that $y^U = x^{\iota(U)}$ for $U \in \text{SSTab}(\lambda)_{\leq r}$. For example, if $\lambda = (1^2)$

, $\mu = (2)$, $T_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$ and $U = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$, then $\iota(U) =$



, $y_1 = x^{T_1} = x_1^2$, $y_2 = x^{T_2} = x_1 x_2$, $y^U = y_1 y_2$ and $x^{\iota(U)} = x_1^3 x_2$.

By Corollary 2.7, in the case of s -variables we have;

$$s_\mu(x_1, \dots, x_s) = \sum_{T \in \text{SSTab}(\mu)_{\leq s}} x^T = y_1 + y_2 + \dots + y_r.$$

Thus by Proposition 2.8, we have;

$$\begin{aligned}
 s_\lambda[s_\mu](x_1, \dots, x_s) &= s_\lambda(x^{T_1}, \dots, x^{T_r}) = s_\lambda(y_1, \dots, y_r) \\
 &= \sum_{U \in \text{SSTab}(\lambda)_{\leq r}} y^U \\
 &= \sum_{\iota(U) \in \text{SSTab}(\lambda[\mu])_{\leq s}} x^{\iota(U)}.
 \end{aligned}$$

Here by taking the limit $s \rightarrow \infty$, we have the following equality as symmetric function;

$$s_\lambda[s_\mu] = \sum_{T \in \text{SSTab}(\lambda[\mu])} x^T = \sum_{\nu \vdash mn} Y_{\lambda[\mu]}^\nu m_\nu.$$

□

4. THE FIRST TERM OF PLETHYSM

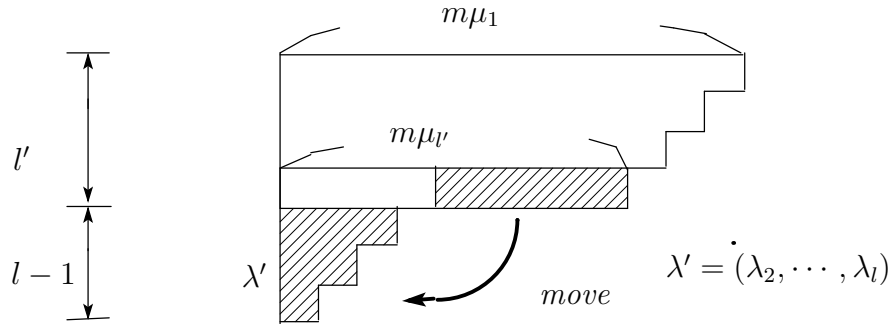
Definition 4.1. Let $\lambda \vdash m$ and $\mu \vdash n$. The first term of the plethysm $s_\lambda[s_\mu]$ is the maximal element in the set $\{\nu \vdash mn \mid a_{\lambda[\mu]}^\nu \neq 0\}$ with respect to the reverse lexicographic order.

Theorem 4.2. ([2; Conjecture 2], [3; Conjecture 1.2] and [12; Conjecture 5.1])

Let $\lambda \vdash m, \mu \vdash n, l = l(\lambda)$ and $l' = l(\mu)$. (Where $l = l(\lambda)$ is the length of partition λ .) Then the first term of plethysm $s_\lambda[s_\mu]$ is

$$\nu_0 := (m\mu_1, m\mu_2, \dots, m(\mu_{l'} - 1) + \lambda_1, \lambda_2, \dots, \lambda_l).$$

Moreover the coefficient of the first term is equal to 1.



Proof. By proposition 2.3, note that

$$s_\nu = \sum_{\kappa} K_{\kappa, \nu}^{-1} h_\kappa.$$

Then we have;

$$\begin{aligned}
a_{\lambda[\mu]}^\nu &= \langle s_\lambda[s_\mu], s_\nu \rangle \\
&= \langle s_\lambda[s_\mu], \sum_{\kappa} K_{\kappa,\nu}^{-1} h_\kappa \rangle \\
&= \sum_{\kappa} K_{\kappa,\nu}^{-1} \langle s_\lambda[s_\mu], h_\kappa \rangle \\
&= \sum_{\kappa} K_{\kappa,\nu}^{-1} Y_{\lambda[\mu]}^\kappa, \quad (\text{by Theorem 3.5 and property of } \langle \cdot, \cdot \rangle) \\
&= Y_{\lambda,\mu}^\nu + \sum_{\kappa > \nu} K_{\kappa,\nu}^{-1} Y_{\lambda[\mu]}^\kappa, \quad (\text{by Corollary 2.6 (ii)})
\end{aligned}$$

Thus the assertion follows from the next lemma.

Lemma 4.3. (1) $\max \{ \nu \vdash mn \mid Y_{\lambda[\mu]}^\nu \neq 0 \} = \nu_0$.
(2) $Y_{\lambda[\mu]}^{\nu_0} = 1$.

Proof. We define a total order on the set of monomials in x_i 's ($i = 1, 2, \dots$) by lexicographic order. (For example, $x_2^3 < x_1 x_2 < x_1^2 x_2 < x_1^3$.) Then we can arrange elements of $\text{SSTab}(\mu)$ according to this order as follows;

$$\begin{array}{ccc}
T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline \vdots & & & \\ \hline l' & l' & & \\ \hline \end{array} & T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline \vdots & & & \\ \hline l' & & & \diagdown l' + 1 \\ \hline \end{array} & T_3 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline \vdots & & & \\ \hline l' & & & \diagdown l' + 2 \\ \hline \end{array} \\
\vdots & & & \\
\dots & T_l = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \\ \hline \vdots & & & \\ \hline l' & & & \diagdown l' + l - 1 \\ \hline \end{array} & &
\end{array}$$

And weight of these are

$$\begin{cases} \text{wt}(T_1) &= (\mu_1, \mu_2, \dots, \mu_{l'}) \\ \text{wt}(T_2) &= (\mu_1, \mu_2, \dots, \mu_{l'} - 1, 1) \\ \text{wt}(T_3) &= (\mu_1, \mu_2, \dots, \mu_{l'} - 1, 0, 1) \\ \vdots &\vdots \\ \text{wt}(T_l) &= (\mu_1, \mu_2, \dots, \mu_{l'} - 1, 0, \dots, 0, 1). \end{cases}$$

Thus the sequence $T^{(1)} \geq T^{(2)} \geq \dots \geq T^{(m)}$ that have a maximal weight under the condition $Y_{\lambda[\mu]}^\nu \neq 0$ is only

$$\begin{cases} T^{(1)} = \dots = T^{(\lambda_1)} & = T_1 \\ T^{(\lambda_1+1)} = \dots = T^{(\lambda_1+\lambda_2)} & = T_2 \\ \vdots & \vdots \\ T^{(m-\lambda_l+1)} = \dots = T^{(m)} & = T_l \end{cases} .$$

Therefore the maximal weight is

$$\begin{aligned} \text{wt}(T^{(1)}) + \dots + \text{wt}(T^{(m)}) &= \lambda_1 \text{wt}(T_1) + \dots + \lambda_l \text{wt}(T_l) \\ &= (m\mu_1, m\mu_2, m(\mu_{l'} - 1) + \lambda_1, \lambda_2, \dots, \lambda_l) \\ &= \nu_0 \end{aligned}$$

□

□

In particular, from the first half of this proof we get a new combinatorial formula for plethysm. (Note that we can also express $K_{\kappa, \nu}^{-1}$ combinatorially.[6])

Corollary 4.4.

$$a_{\lambda[\mu]}^\nu = \sum_{\kappa \vdash mn} K_{\kappa, \nu}^{-1} Y_{\lambda[\mu]}^\kappa .$$

Example 4.5. We calculate $a_{(2)[(2)]}^{(2,2)}$ from this formula. Since $K_{(2,2),(2,2)}^{-1} = 1$, $K_{(3,1),(2,2)}^{-1} = -1$, $K_{(4),(2,2)}^{-1} = 0$, $Y_{(2)[(2)]}^{(2,2)} = 2$ and $Y_{(2)[(2)]}^{(3,1)} = 1$, we have

$$a_{(2)[(2)]}^{(2,2)} = K_{(2,2),(2,2)}^{-1} Y_{(2)[(2)]}^{(2,2)} + K_{(3,1),(2,2)}^{-1} Y_{(2)[(2)]}^{(3,1)} + K_{(4),(2,2)}^{-1} Y_{(2)[(2)]}^{(4)} = 2 - 1 + 0 = 1.$$

5. SOME REMARKS

Our purpose for plethysm is stated as follows.

Problem. Express the expansion coefficients $a_{\lambda[\mu]}^\nu$ combinatorially.

For this problem, by imitating a proof of Littlewood-Richardson rule given in [7], we have the followings.

Let $l = l(\nu)$. By Jacobi-Trudi's formula, we have

$$s_\nu = \sum_{\pi \in S_l} \text{sgn}(\pi) h_{\pi * \nu} \quad , \quad (\pi * \nu = (\nu_{\pi(i)} - \pi_i + i)_{1 \leq i \leq l}) .$$

Here by Proposition 3.5, we have

$$\begin{aligned}
a_{\lambda[\mu]}^\nu &= \langle s_\lambda[s_\mu], s_\nu \rangle \\
&= \sum_{\pi \in S_l} \text{sgn}(\pi) \langle s_\lambda[s_\mu], h_{\pi * \nu} \rangle \\
&= \sum_{\pi \in S_l} \text{sgn}(\pi) Y_{\lambda[\mu]}^{\pi * \nu} .
\end{aligned}$$

So set $A = \{(\pi, T) \mid \pi \in S_l, T \in \text{SSTab}(\lambda[\mu]; \pi * \nu)\}$, then we have

$$a_{\lambda[\mu]}^\nu = \sum_{(\pi, T) \in A} \text{sgn}(\pi) .$$

Therefore the following conjecture is expected.

Conjecture 5.1. *There are a subset $S_0 \subset \text{SSTab}(\lambda[\mu])$ and a bijective map $\phi: A - A_0 \ni (\pi, T) \rightarrow (\pi', T') \in A - A_0$ such that $\text{sgn}(\pi) = -\text{sgn}(\pi')$, where $A_0 = \{(\pi, T) \in A \mid T \in S_0\}$.*

In the case of Littlewood-Richardson rule, it is possible to take "the set of lattice permutations" as A_0 . Then ϕ can be defined "properly". ([7])

Indeed, the following property holds.

Lemma 5.2. *For any $\lambda \vdash m, \mu \vdash n$ and $\nu \vdash mn$, we have*

$$Y_{\lambda[\mu]}^\nu \geq a_{\lambda[\mu]}^\nu .$$

Proof. Recall

$$s_\lambda[s_\mu] = \sum_{\kappa \vdash mn} a_{\lambda[\mu]}^\kappa s_\kappa ,$$

and comparing the coefficients of the monomial symmetric function m_ν ,

we have

$$\begin{aligned}
Y_{\lambda[\mu]}^\nu &= \sum_{\kappa \vdash mn} a_{\lambda[\mu]}^\kappa K_{\kappa, \nu} \\
&= a_{\lambda[\mu]}^\nu + \sum_{\kappa > \nu} a_{\lambda[\mu]}^\kappa K_{\kappa, \nu} , \quad (\text{by Corollary 2.6 (i) and } K_{\nu, \nu} = 1) \\
&\geq a_{\lambda[\mu]}^\nu , \quad (a_{\lambda[\mu]}^\kappa \geq 0 \text{ and } K_{\kappa, \nu} \geq 0).
\end{aligned}$$

□

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